LK Mathematik * K13 * Integrationsverfahren

1. Schreiben Sie integralfrei durch Anwendung eines geeigneten Integrationsverfahrens. Geben Sie auch den jeweiligen Definitionsbereich der Integralfunktion an!

a) $\int_0^x \frac{2t+1}{t+1}\,dt$ b) $\int_1^x \frac{3t}{\sqrt{t^2+1}}\,dt$ c) $\int_0^x \frac{t+3}{t^2-4}\,dt$

d) $\int_1^x (2t-3)\cdot e^{5t}\,dt$ e) $\int_1^x \frac{\ln(2t)}{5t}\,dt$ f) $\int_1^x \frac{e^t}{e^t+e^{-t}}\,dt$

2. Ermitteln Sie mit geeignetem Verfahren das unbestimmte Integral!

a) $\int x\cdot\cos(x)\,dx$ b) $\int 5x\cdot e^{-x^2}\,dx$ c) $\int \frac{1}{\sqrt{x}\cdot(1+\sqrt{x})^2}\,dx$

d) $\int \frac{1}{x\cdot(\ln(x))^2}\,dx$ e) $\int \frac{2}{x^2-x}\,dx$ f) $\int 2x^2\cdot\sqrt{1+5x^3}\,dx$

3. Berechnen Sie mit geeignetem Verfahren!

a) $\int_0^{2\pi} (\sin(x))^2\,dx$ b) $\int_{-1}^0 \frac{4}{3+2x}\,dx$ c) $\int_{-1}^0 \frac{x}{\sqrt{3+2x^2}}\,dx$

d) $\int_0^e \frac{\ln(2x+2)}{x+1}\,dx$ e) $\int_0^1 x\cdot e^{1-x^2}\,dx$ f) $\int_0^5 \frac{2}{\sqrt{4+x}}\,dx$

4. Erklären Sie das (scheinbare) Paradoxon der partiellen Integration:

$$\int \frac{1}{x}\,dx = \int 1\cdot\frac{1}{x}\,dx = x\cdot\frac{1}{x} - \int x\cdot\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{1}{x}\,dx$$

also $0 = 1$????

5. Es gelte $f_k(x) = \frac{e^{kx}+e^{-kx}}{2k}$ mit $k \in \mathbb{R}^+$ und $F_k(x) = \int_0^x f_k(t)\,dt$

Ermitteln Sie eine integralfreie Darstellung von $F_1(x)$ und zeigen Sie die Gültigkeit der Beziehung $(f_1(x))^2 = 1 + (F_1(x))^2$ (aus Abi 2003 / I, 1).

G. Rasch, Januar 2008

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1. a) $\int \frac{2t+1}{t+1}\,dt = \int \frac{2z-1}{z}\,dz = 2z - \ln|z| + C = 2(t+1) - \ln|t+1| + C \quad (\text{Subst.: } z = t+1)$

$\int_0^x \frac{2t+1}{t+1}\,dt = 2(x+1) - \ln|x+1| - (2\cdot 1 - \ln(1)) = 2x - \ln(x+1) \quad \text{mit } x \in \,]-1;\infty[$

b) $\int \frac{3t}{\sqrt{t^2+1}}\,dt = \int \frac{3}{2\sqrt{z}}\,dz = 3\cdot\sqrt{z} + C = 3\cdot\sqrt{t^2+1} + C \quad (\text{Subst.: } z = t^2+1)$

$\int_1^x \frac{3t}{\sqrt{t^2+1}}\,dt = 3\cdot\sqrt{x^2+1} - 3\cdot\sqrt{2} \quad \text{mit } x \in R$

c) $\int \frac{t+3}{t^2-4}\,dt = \int \frac{t+3}{(t-2)\cdot(t+2)}\,dt = \int \frac{1,25}{t-2} - \frac{0,25}{t+2}\,dt = \frac{5}{4}\cdot\ln|t-2| - \frac{1}{4}\ln|t+2| + C$

$\int_0^x \frac{t+3}{t^2-4}\,dt = \frac{5}{4}\cdot\ln|x-2| - \frac{1}{4}\ln|x+2| - \left(\frac{5}{4}\ln(2) - \frac{1}{4}\ln(2)\right) =$

$\frac{5}{4}\cdot\ln|x-2| - \frac{1}{4}\ln|x+2| - \ln(2) \quad \text{mit } x \in \,]-2\,;2\,[$

d) $\int (2t-3)\cdot e^{5t}\,dt = (2t-3)\cdot\frac{1}{5}e^{5t} - \int 2\cdot\frac{1}{5}e^{5t}\,dt = \ldots = (0,4t - 0,68)\cdot e^{5t} + C \quad (\text{part. Int.})$

$\int_1^x (2t-3)\cdot e^{5t} = (0,4x - 0,68)\cdot e^{5x} - (0,4 - 0,68)\cdot e^5 = (0,4x - 0,68)\cdot e^{5x} + 0,28\cdot e^5$

mit $x \in R$

e) $\int \frac{\ln(2t)}{5t}\,dt = \int \frac{z}{5}\,dz = 0,10\cdot z^2 + C = 0,10\cdot(\ln(2t))^2 + C$

$\int_1^x \frac{\ln(2t)}{5t}\,dt = 0,10\cdot(\ln(2x))^2 - 0,10\cdot(\ln(2))^2 \quad \text{mit } x \in R^+$

f) $\int \frac{e^t}{e^t+e^{-t}}\,dt = \int \frac{e^{2t}}{e^{2t}+1}\,dt = \int \frac{1}{2}\cdot\frac{1}{z}\,dz = \frac{1}{2}\cdot\ln|z| + C = \frac{1}{2}\cdot\ln\left(e^{2t}+1\right) + C$

$\int_1^x \frac{e^t}{e^t+e^{-t}}\,dt = \frac{1}{2}\cdot\ln\left(e^{2x}+1\right) - \frac{1}{2}\cdot\ln\left(e^2+1\right) \quad \text{mit } x \in R$

2. a) $\int x\cdot\cos(x)\,dx = x\cdot\sin(x) - \int 1\cdot\sin(x)\,dx = x\cdot\sin(x) + \cos(x) + C \quad (\text{part. Int.})$

b) $\int 5x\cdot e^{-x^2}\,dx = -\frac{5}{2}\cdot\int e^z\,dz = -2,5\cdot e^z + C = -2,5\cdot e^{-x^2} + C \quad (\text{Subst.: } z = -x^2)$

c) $\int \frac{1}{\sqrt{x}\cdot(1+\sqrt{x})^2}\,dx = \int \frac{2}{z^2}\,dz = -\frac{2}{z} + C = -\frac{2}{1+\sqrt{x}} + C \quad (\text{Subst.: } z = 1+\sqrt{x})$

d) $\int \frac{1}{x\cdot(\ln(x))^2}\,dx = \int \frac{1}{z^2}\,dz = -\frac{1}{z} + C = -\frac{1}{\ln(x)} + C \quad (\text{Subst.: } z = \ln(x))$

e) $\int \frac{2}{x^2-x}\,dx = \int \frac{-2}{x} + \frac{2}{x-1}\,dx = -2\ln|x| + 2\ln|x-1| + C = 2\ln\left|\frac{x-1}{x}\right| + C$

f) $\int 2x^2\cdot\sqrt{1+5x^3}\,dx = \frac{2}{15}\int\sqrt{z}\,dz = \frac{2}{15}\cdot\frac{2}{3}z^{3/2} + C = \frac{4}{45}\cdot\left(1+5x^3\right)^{3/2} + C$

$(\text{Subst.: } z = 1+5x^3)$

3. a) $\int (\sin(x))^2\,dx = \sin(x)\cdot(-\cos(x)) - \int \cos(x)\cdot(-\cos(x))\,dx =$

$-\sin(x)\cdot\cos(x) + \int (\cos(x))^2\,dx = -\sin(x)\cdot\cos(x) + \int 1-(\sin(x))^2\,dx =$

$-\sin(x)\cdot\cos(x) + x - \int (\sin(x))^2\,dx \quad \Rightarrow$

$\int (\sin(x))^2\,dx = \frac{1}{2}\cdot\left(x - \sin(x)\cdot\cos(x)\right) + C$

$\int_0^{2\pi} (\sin(x))^2\,dx = \frac{1}{2}\cdot(2\pi-0) - \frac{1}{2}\cdot(0-0) = \pi$

b) $\int \frac{4}{3+2x}\,dx = 2\cdot\ln|3+2x| + C \;\Rightarrow\; \int_{-1}^{0} \frac{4}{3+2x}\,dx = 2\cdot\ln(3) - 2\cdot\ln(1) = 2\cdot\ln(3)$

c) $\int \frac{x}{\sqrt{3+2x^2}}\,dx = \int \frac{1}{4\cdot\sqrt{z}}\,dz = \frac{1}{2}\cdot\sqrt{z} + C = \frac{1}{2}\cdot\sqrt{3+2x^2} + C \quad (\text{Subst.: } z = 3+2x^2)$

$\int_{-1}^{0} \frac{x}{\sqrt{3+2x^2}}\,dx = \frac{1}{2}\cdot\sqrt{3+0} - \frac{1}{2}\cdot\sqrt{3+2} = \frac{1}{2}\cdot\left(\sqrt{3}-\sqrt{5}\right)$

d) $\int \frac{\ln(2x+2)}{x+1}\,dx = \int z\,dz = 0{,}5\cdot z^2 + C = 0{,}5\cdot\left(\ln(2x+2)\right)^2 + C \quad (\text{Subst.: } z=\ln(2x+2))$

$\int_0^{e} \frac{\ln(2x+2)}{x+1}\,dx = 0{,}5\cdot\left(\ln(2e+2)\right)^2 - 0{,}5\cdot(\ln(2))^2$

e) $\int x\cdot e^{1-x^2}\,dx = -\frac{1}{2}\int e^z\,dz = -\frac{1}{2}\cdot e^z + C = -\frac{1}{2}\cdot e^{1-x^2} + C \quad (\text{Subst.: } z = 1-x^2)$

$\int_0^{1} x\cdot e^{1-x^2}\,dx = -\frac{1}{2}\cdot e^0 + \frac{1}{2}\cdot e^1 = \frac{1}{2}\cdot(e-1)$

f) $\int \frac{2}{\sqrt{4+x}}\,dx = \int \frac{2}{\sqrt{z}}\,dz = 4\cdot\sqrt{z} + C = 4\cdot\sqrt{4+x} + C \quad (\text{Subst.: } z = 4+x)$

$\int_0^{5} \frac{2}{\sqrt{4+x}}\,dx = 4\cdot\sqrt{4+5} - 4\cdot\sqrt{4+0} = 12 - 8 = 4$

4. $\int \frac{1}{x}\,dx = \int 1\cdot\frac{1}{x}\,dx = x\cdot\frac{1}{x} - \int x\cdot\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{1}{x}\,dx$ ist korrekt, denn

$\int \frac{1}{x}\,dx = \ln|x| + C_1$ und $1 + \int \frac{1}{x}\,dx = 1 + \ln|x| + C_2$ also $C_1 = 1+C_2$.

5. $\int f_k(x)\,dx = \int \frac{e^{kx}+e^{-kx}}{2k}\,dx = \frac{e^{kx}}{2k^2} - \frac{e^{-kx}}{2k^2} + C \;\Rightarrow$

$F_k(x) = \int_0^{x} f_k(t)\,dt = \frac{e^{kx}}{2k^2} - \frac{e^{-kx}}{2k^2} - \left(\frac{e^0}{2k^2} - \frac{e^0}{2k^2}\right) = \frac{e^{kx}-e^{-kx}}{2k^2}$

$F_1(x) = \frac{e^x - e^{-x}}{2}$ und $f_1(x) = \frac{e^x + e^{-x}}{2}$ also $\left(f_1(x)\right)^2 = \frac{e^{2x}+2+e^{-2x}}{4}$ und

$1 + \left(F_1(x)\right)^2 = 1 + \frac{e^{2x}-2+e^{-2x}}{4} = \frac{e^{2x}+2+e^{-2x}}{4} = \left(f_1(x)\right)^2$ q.e.d.