Jahrgangsstufe 11 * Komplexe Zahlen

Gleichungen in C, die sich auf quadratische und reine Gleichungen zurückführen lassen

1. $3z^{16} = 2z^{8} + 8$
2. $z^{3} + 3 = \dfrac{4}{z^{3}}$
3. $z^{18} = -i\,z^{9}$
4. $z^{8} = 15z^{4} + 16$
5. $z^{3} = 8z^{-3} + 7$
6. $z^{10} + 2 = 2z^{5}$

Und nun noch zwei rechenaufwändige Gleichungen

7. $z^{3} + 1 = \dfrac{i}{z^{3}} + i$
8. $z^{8} + 4 = (4 + 2i)z^{4} - 4i$

Lösungen:

1. $z_k = \sqrt[8]{2}\,E(k\cdot 45^o)$ *mit* $k = 0,1,2,...,7$ und
   $z_n = \sqrt[8]{\dfrac{4}{3}}\,E(22{,}5^o + n\cdot 45^o)$ *mit* $n = 0,1,2,...,7$
2. $z_1 = 1$; $z_2 = E(120^o)$; $z_3 = E(240^o)$; $z_4 = \sqrt[3]{4}\,E(60^o)$; $z_5 = \sqrt[3]{4}\,E(180^o)$; $z_6 = \sqrt[3]{4}\,E(300^o)$
3. $z_k = E(30^0 + k\cdot 40^o)$ *mit* $k = 0,1,2,...,8$ *sowie* $z_9 = 0$
4. $z_1 = 2$; $z_2 = 2i$; $z_3 = -2$; $z_4 = -2i$; $z_5 = E(45^0)$; $z_6 = E(135^o)$; $z_7 = E(225^o)$; $z_8 = E(315^o)$
5. $z_1 = 2$; $z_2 = 2E(120^o)$; $z_3 = 2E(240^o)$; $z_4 = E(60^o)$; $z_5 = E(180^o) = -1$; $z_6 = E(300^o)$
6. $z_k = \sqrt[10]{2}\,E(9^o + k\cdot 72^o)$ *bzw.* $z_n = \sqrt[10]{2}\,E(63^o + n\cdot 72^o)$ *mit* $k, n = 0, 1, 2, ..., 4$
7. $z_1 = E(30^o)$; $z_2 = E(150^o)$; $z_3 = E(270^o) = -i$; $z_4 = E(60^o)$; $z_5 = E(180^o) = -1$; $z_6 = E(300^o)$
8. $z_1 = \sqrt[4]{2}$; $z_2 = \sqrt[4]{2}\,i$; $z_3 = -\sqrt[4]{2}$; $z_4 = -\sqrt[4]{2}\,i$;
   $z_5 = \sqrt[8]{8}\,E(11{,}25^o)$; $z_6 = \sqrt[8]{8}\,E(101{,}25^o)$; $z_7 = \sqrt[8]{8}\,E(191{,}25^o)$; $z_8 = \sqrt[8]{8}\,E(281{,}25^o)$

Ausführliche Lösungen auf der Rückseite!

Ausführliche Lösungen:

1. $\tilde{z} = z^8$; $3\tilde{z}^2 - 2\tilde{z} - 8 = 0 \Leftrightarrow \tilde{z}^2 - \frac{2}{3}\tilde{z} - \frac{8}{3} = 0 \Leftrightarrow (\tilde{z}-2)\cdot(\tilde{z}+\frac{4}{3}) = 0 \Leftrightarrow$

$\Leftrightarrow z^8 = 2$ *bzw.* $z^8 = -\frac{4}{3} \Leftrightarrow z^8 = 2E(0^o)$ *bzw.* $z^8 = \frac{4}{3}E(180^o)$

$z^8 = 2E(0^o)$ *hat die Lösungen* $z_k = \sqrt[8]{2}\,E(k\cdot 45^o)$ *mit* $k = 0, 1, 2, ..., 7$

$z^8 = \frac{4}{3}\,E(180^o)$ *hat die Lösungen* $z_n = \sqrt[8]{\frac{4}{3}}\,E(22{,}5^o + n\cdot 45^o)$ *mit* $n = 0, 1, 2, ... ,7$

2. $z^3 + 3 = \frac{4}{z^3} \Leftrightarrow z^6 + 3z^3 - 4 = 0 \Leftrightarrow u^2 + 3u - 4 = 0$ *mit* $u = z^3$

$(u+4)\cdot(u-1) = 0 \Leftrightarrow u_1 = 1$; $u_2 = -4$ *also* $z^3 = 1E(0^o)$ *bzw.* $z^3 = 4E(180^o)$

$z^3 = 1E(0^o)$ *hat die Lösungen* $z_1 = 1$; $z_2 = E(120^o)$; $z_3 = E(240^o)$

$z^3 = 4E(180^o)$ *hat die Lösungen* $z_4 = \sqrt[3]{4}\,E(60^o)$; $z_5 = \sqrt[3]{4}\,E(180^o)$; $z_6 = \sqrt[3]{4}\,E(300^o)$

3. $z^{18} = -i\,z^9 \Leftrightarrow z^9\,(z^9 + i) = 0 \Leftrightarrow z^9 = 0$ *oder* $z^9 = -i = E(270^o)$

also $z_k = E(30^0 + k\cdot 40^o)$ *mit* $k = 0,1,2,...,8$ *sowie* $z_9 = 0$

4. $z^8 = 15z^4 + 16 \Leftrightarrow u^2 - 15u - 16 = 0$ *mit* $u = z^4 \Leftrightarrow (u-16)\cdot(u+1) = 0$

also $z^4 = 16$ *bzw.* $z^4 = -1 = E(180^o)$; die Lösungen lauten damit

$z_1 = 2$; $z_2 = 2i$; $z_3 = -2$; $z_4 = -2i$; $z_5 = E(45^0)$; $z_6 = E(135^o)$; $z_7 = E(225^o)$; $z_8 = E(315^o)$

5. $z^3 = 8z^{-3} + 7 \Leftrightarrow z^6 - 7z^3 - 8 = 0 \Leftrightarrow u^2 - 7u - 8 = 0$ *mit* $u = z^3 \Leftrightarrow$

$(u-8)\cdot(u+1) = 0$ *also* $z^3 = 8$ *oder* $z^3 = -1 = E(180^o)$; die Lösungen lauten also

$z_1 = 2$; $z_2 = 2E(120^o)$; $z_3 = 2E(240^o)$; $z_4 = E(60^o)$; $z_5 = E(180^o) = -1$; $z_6 = E(300^o)$

6. $z^{10} + 2 = 2z^5 \Leftrightarrow u^2 - 2u + 2 = 0$ *mit* $u = z^5 \Leftrightarrow (u-1)^2 = -1$ *also* $u_{1/2} = 1 \pm i$

$z^5 = \sqrt{2}\,E(45^o)$ *oder* $z^5 = \sqrt{2}\,E(315^o)$ *ist also zu lösen*; die Lösungen lauten damit

$z_k = \sqrt[10]{2}\,E(9^o + k\cdot 72^o)$ *bzw.* $z_n = \sqrt[10]{2}\,E(63^o + n\cdot 72^o)$ *mit* $k, n = 0, 1, 2, ..., 4$

7. $z^3 + 1 = \frac{i}{z^3} + i \Leftrightarrow z^6 + (1-i)z^3 - i = 0 \Leftrightarrow u^2 + (1-i)u - i = 0$ *mit* $u = z^3$

$u^2 + (1-i)u - i = 0 \Leftrightarrow (u + \frac{1}{2}(1-i))^2 = \frac{1}{2}i \Leftrightarrow u_{1/2} = -\frac{1}{2}(1-i) \pm \frac{1}{\sqrt{2}}E(45^o)$

$u_{1/2} = -\frac{1}{2}(1-i) \pm \frac{1}{2}(1+i) \Leftrightarrow u_1 = i$; $u_2 = -1$ *also* $z^3 = E(90^o)$ *bzw.* $z^3 = E(180^o)$

$z_1 = E(30^o)$; $z_2 = E(150^o)$; $z_3 = E(270^o) = -i$; $z_4 = E(60^o)$; $z_5 = E(180^o) = -1$; $z_6 = E(300^o)$

8. $z^8 + 4 = (4+2i)z^4 - 4i \Leftrightarrow u^2 - (4+2i)u + 4 + 4i = 0$ *mit* $u = z^4$

$u^2 - (4+2i)u + 4 + 4i = 0 \Leftrightarrow [u-(2+i)]^2 - (2+i)^2 + 4 + 4i = 0 \Leftrightarrow$

$[u-(2+i)]^2 = -1 \Leftrightarrow [u-(2+i)]^2 = E(180^o) \Leftrightarrow u-(2+i) = \pm i$

$u_1 = 2$; $u_2 = 2+2i = 2\sqrt{2}\,E(45^o)$ *also* $z^4 = 2$ *oder* $z^4 = 2\sqrt{2}\,E(45^o)$

$z_1 = \sqrt[4]{2}$; $z_2 = \sqrt[4]{2}\,i$; $z_3 = -\sqrt[4]{2}$; $z_4 = -\sqrt[4]{2}\,i$;

$z_5 = \sqrt[8]{8}\,E(11{,}25^o)$; $z_6 = \sqrt[8]{8}\,E(101{,}25^o)$; $z_7 = \sqrt[8]{8}\,E(191{,}25^o)$; $z_8 = \sqrt[8]{8}\,E(281{,}25^o)$